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Covariant Self-force Regularization of a Particle Orbiting a Schwarzschild Black Hole

- Mode Decomposition Regularization -

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abstract

Covariant structure of the self-force of a particle in a general curved background has been made clear in the cases of scalar [1], electromagnetic [2], and gravitational charges [3,4]. Namely, what we need is the part of the self-field that is non-vanishing off and within the past light-cone of particle's location, the so-called tail. The radiation reaction force in the absence of external fields is entirely contained in the tail. In this paper, we develop mathematical tools for the regularization and propose a practical method to calculate the self-force of a particle orbiting a Schwarzschild black hole.

I. INTRODUCTION

For a particle carrying a scalar, electromagnetic or gravitational charge, the field configuration of the corresponding type varies in time as it moves around a black hole. To the lowest order in the charge, the particle motion follows a geodesic in the black hole background in the absence of external force fields. However, a part of the time-varying field becomes radiation near the future null infinity or future horizon and carries the energy-momentum away from the system, and a part of it is scattered by the background curvature and comes back to the location of the particle. Hence the motion of the particle is affected in the next order. The force exerted by the back-scattered self-field is called the local reaction force or simply the self-force. To establish a calculational strategy of this force is our ultimate goal.

It is noted that we may consider the local reaction force to consist of the two parts: The part that describes the loss of the energy-momentum of the particle and the other part that only contributes to the shift of conserved quantities. In the case of geodesic motion in the Schwarzschild background, this division is unambiguous because the geodesic motion is completely determined by the two conserved quantities; the energy and the z-component of the angular momentum (a geodesic can be assumed to be on the equatorial plane without loss of generality). In the case of the Kerr background, however, it seems unclear if the two parts can be identified uniquely because of the presence of the conserved quatity called the Carter constant which has no explicit relation to the energy-momentum of the system.

When we attempt to calculate the reaction force on a point charge (particle), we encounter the divergence of the force. Hence, it is necessary to extract out the physically meaningful finite part of the force. Since the force is a vector by definition with respect to a background space-time, and any vector depends on the choice of coordinates in a covariant manner, the finite reaction force should be given covariantly.

The covariant structure of the reaction force was investigated in the scalar case in [1], in the electromagnetic case in [2], and in the gravitational case in [3,4]. In these investigations, the divergent part of the force was found to be described solely with the local geometrical quantities, whereas the finite part that contributes to the equation of motion was found to be given by the tail part which is due to the curvature scattering of the self-field.

Since the tail part depends non-locally on the geometry of the background spacetime, it is almost impossible to calculate it directly. However, for a certain class of spacetimes such as Schwarzschild/Kerr geometries, there is a way to calculate the full field generated by a point charge. Considering a field point slightly off the particle trajectory, it is then possible to obtain the tail part by subtracting the locally given divergent part from the full field. Thus denoting the field by ${}_s\phi$ for the scalar ($s=0$), electromagnetic ($s=1$) or gravitational ($s=2$) case, with its spacetime indices suppressed, the reaction force is schematically given by

$$F_\alpha(\tau_0) = \lim_{x\to z(\tau_0)} F_\alpha[{}_s\phi^{\rm tail}](x)\,, \tag{1.1}$$

$$F_\alpha[{}_s\phi^{\rm tail}](x) = F_\alpha[{}_s\phi^{\rm full}](x) - F_\alpha[{}_s\phi^{\rm dir}](x)\,, \quad (x \neq z(\tau))\,, \tag{1.2}$$

where z is the orbit of the particle with the proper time τ, and τ_0 is the proper time at the orbital point at which we calculate the force. The symbol ${}_s\phi^{\rm tail}$ stands for the tail field induced by the particle which is regular in the coincidence limit $x\to z(\tau)$, ${}_s\phi^{\rm full}$ for the full field, and ${}_s\phi^{\rm dir}$ for the direct part as defined in Refs. [1–4]. Both ${}_s\phi^{\rm full}$ and ${}_s\phi^{\rm dir}$ diverge in the coincidence limit $x\to z(\tau)$. $F_\alpha[...]$ is a tensor operator on the field, and is defined as

$$F_\alpha[{}_s\phi] = \begin{cases} qP_\alpha{}^\beta\nabla_\beta\phi & (s=0),\\ eP_\alpha{}^\beta(\phi_{\gamma;\beta}-\phi_{\beta;\gamma})V^\gamma & (s=1),\\ -mP_\alpha{}^\beta\left(\phi_{\beta\gamma;\delta}-\frac{1}{2}g_{\beta\gamma}\phi^\epsilon{}_{\epsilon;\delta}-\frac{1}{2}\phi_{\gamma\delta;\beta}+\frac{1}{4}g_{\gamma\delta}\phi^\epsilon{}_{\epsilon;\beta}\right)V^\gamma V^\delta & (s=2), \end{cases} \tag{1.3}$$

where $P_\alpha{}^\beta = \delta_\alpha{}^\beta + V_\alpha V^\beta$ is the projection tensor with V^α being an appropriate extension of the four velocity $v^\alpha(\tau_0)$ off the orbital point.

In practice, it is a non-trivial task to perform the subtraction of the direct part, which we call the '*Subtraction Problem*'. In this paper we propose a method to carry out this subtraction procedure covariantly.

It should be noted, however, that solving the subtraction problem is not enough when one deals with the gravitational case. In the scalar or electromagnetic case, the reaction force is a gauge-invariant notion. In contrast, the reaction force in the gravitational case does depend on the gauge choice. Therefore, one has to fix the gauge appropriately and evaluate the full metric perturbation and its direct part in the same gauge before calculating the force. We call this the '*Gauge Problem*', which seems to be a very difficult problem to solve. We do not discuss the possible solution of the gauge problem in this paper, but leave it for future work. Instead, we only calculate the direct part of the linear gravitational force under the harmonic gauge condition.

In this paper, as a first step, we consider the case that the background is approximated by the Schwarzschild blackhole and use the Boyer-Lindquist coordinates,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\,. \tag{1.4}$$

We use the notation that $x=\{t,\, r,\, \theta,\, \phi\}$ stands for a field point, and $z(\tau_0)=z_0=\{t_0,\, r_0,\, \theta,\, \phi\}$ for an orbital point.

The paper is organized as follows. In Section II, we describe basics of the issue. In Section III, we describe our strategy of regularization, which we call the 'mode decomposition regularization'. In Section IV, focusing on the scalar case, we present our method for the mode-decomposition of the direct part of the self-force. In doing so, we obtain a useful mathematical formula, Eq. (4.30), for the harmonic decomposition. In Section V, summing over the azimuthal modes m, we compare our result with Barack and Ori [15,16]. We find complete agreement between the two. In Section VI, we discuss the pros and cons and conclude the paper by pointing out some future issues. Technical details as well as the mode decomposition regularization of the electromagnetic and gravitational cases are differed in Appendices.

II. BASICS

There are two important issues in the regularization. One is how to implement a regularization method in Eq. (1.1). Both ${}_s\phi^{\rm full}(x)$ and ${}_s\phi^{\rm dir}(x)$ diverge in the coincidence limit $x\to z_0$. Such divergent quantities are difficult to treat in actual calculations, particularly in numerical computations. We discuss this issue in Subsection II A. The other is how to evaluate the direct part of the field, which we discuss in Subsection II B.

A. Infinite series expansion

We consider the case of a particle in geodesic orbit on the Schwarzschild spacetime. Even in the Newtonian limit, the integration of the orbit involves an elliptic function. Thus, numerical computaions will be necessary at some stage of deriving the self-force. However, we have divergence to be regularized which is difficult to treat numerically. The idea to overcome this difficulty is to replace the divergence by an infinite series, each term of which is finite.

Suppose we have a unique decomposition method applicable to $F_\alpha[{}_s\phi^{\rm full}](x)$, $F_\alpha[{}_s\phi^{\rm dir}](x)$ and $F_\alpha[{}_s\phi^{\rm tail}](x)$ as

$$F_\alpha[{}_s\phi^{\rm full}](x) = \sum_A F^A_\alpha[{}_s\phi^{\rm full}](x)\,, \tag{2.1}$$

$$F_\alpha[{}_s\phi^{\rm dir}](x) = \sum_A F^A_\alpha[{}_s\phi^{\rm dir}](x)\,, \tag{2.2}$$

$$F_\alpha[{}_s\phi^{\rm tail}](x) = \sum_A F^A_\alpha[{}_s\phi^{\rm tail}](x)\,. \tag{2.3}$$

Because of the uniqueness of the decomposition, we have

$$F^A_\alpha[{}_s\phi^{\rm tail}](x) = F^A_\alpha[{}_s\phi^{\rm full}](x) - F^A_\alpha[{}_s\phi^{\rm dir}](x)\,. \tag{2.4}$$

At this stage, we assume that each term of the infinite series is finite, then it is possible take the coincidence limit $x \to z_0$ since $F^A_\alpha[{}_s\phi^{\rm tail}](x)$ is guaranteed to be finite. Therefore we have

$$F_\alpha(\tau_0) = \sum_A F^A_\alpha[{}_s\phi^{\rm tail}](z_0)\,. \tag{2.5}$$

This approach itself does not justify a numerical method in the regularization calculation. However, because of the convergence in the infinite sum (2.5), we can expect that the sum of a finite number of terms in (2.5) gives us an approximated value of the self-force $F_\alpha(\tau_0)$.

However, there is a very delicate problem in this approach. The exact decomposition calculation usually needs the global analytic structure of the field so that we can uniquely define each term in the infinite series. On the other hand, the regularization scheme is derived just by the local analysis of the field [1–4]. Thus the direct part is defined only in the local neighborhood of the particle, and we have an ambiguity in the definition of the direct part. Because of this ambiguity, each term in the infinite series expansion is no more unique but depends on a global extension of the direct part we adopt. Nevertheless, the final result of the self-force should be unique.

In this paper, we present a decomposition method based on the spherical harmonic series expansion. Although we have no explicit proof for the uniqueness of the resulting regularization counter terms for the self-force, the fact that our result completely agrees with that of Barack and Ori [15,16] strongly supports the validity of our method.

B. Direct Part of the Scalar Field

The derivation of the direct part ${}_s\phi^{\rm dir}(x)$ is one of the main issues in the regularization calculation. The direct part of the scalar field is obtained by integrating the direct part of the retarded Green function with the source charge. Here we focus on the scalar case. The electromagnetic and gravitational cases are treated in the same manner, details of which are given in Appendix B.

The direct part of the retarded Green function $G^{\rm dir}$ is given in a covariant manner as

$$G^{\rm dir}(x,x') = -\frac{1}{4\pi}\,\theta[\Sigma(x),x']\sqrt{\Delta(x,x')}\delta\big(\sigma(x,x')\big)\,, \tag{2.6}$$

where $\sigma(x,x')$ is the bi-scalar of half the squared geodesic distance, $\Delta(x,x')$ is the generalized van Vleck-Morette determinant, $\Sigma(x)$ is an arbitrary spacelike hypersurface containing x, and $\theta[\Sigma(x),x'] = 1-\theta[x',\Sigma(x)]$ is equal to 1 when x' lies in the past of $\Sigma(x)$ and vanishes when x' lies in the future. We summarize the basic properties of the bi-scalars $\sigma(x,x')$ and $\Delta(x,x')$ in Appendix A.

The physical meaning of the direct part is understood by the factor $\theta[\Sigma(x),x']\delta\big(\sigma(x,x')\big)$ in Eq. (2.6). Since $\sigma(x,x')$ describes the geodesic distance between x and x', the direct part of the Green function becomes non-zero only when x' lies on the past lightcone of x. Hence the direct part describes the effect of the waves propagated directly from x' to x without scattered by the background curvature.

For the actual evaluation of the direct part, several methods have been proposed. In Ref. [7,9], the direct part of the field is calculated by picking up a limiting contribution in the full Green function from the light cone as

$$\phi^{\rm dir}(x) = \lim_{\epsilon\to+0}\int_{\tau_{\rm ret}(x)-\epsilon}^{\infty} d\tau\, G^{\rm full}(x,z(\tau))S(\tau)\,, \tag{2.7}$$

where $G^{\rm full}$ is the retarded Green function, $S(\tau)$ is the scalar charge density, and $\tau_{\rm ret}(x)$ is the retarded time defined by the past light cone condition of the field point x as

$$\theta[\Sigma(x),z(\tau_{\rm ret})]\delta\big(\sigma(x,z(\tau_{\rm ret}))\big) = 0\,. \tag{2.8}$$

A number of works have been made along this approach [7,9]. However, the caluculation seems rather cumbersome when we apply this method to a general orbit.

In Ref. [6], the direct part was evaluated using the local bi-tensor expansion technique. Using the bi-tensor, the direct part is expanded around the particle location as

$$\phi^{\rm dir}(x) = q\left[\frac{1}{\sigma_{;\alpha}(x,z(\tau_{\rm ret}))v^{\alpha}(\tau_{\rm ret})}\right] + O(y^2)\,, \tag{2.9}$$

where the letters μ, ν, $\cdots$ are used for the indices of the field point x, α, β, $\cdots$ for the indices of the orbital point z, and $v^{\alpha}(\tau)$ is the orbital four velocity at $z(\tau)$. The order of the local expansion is represented by powers of y which is linear to the coordinate difference between the field point x and the orbital point z_0. Because the full force is quadratically divergent, we must carry out the local bi-tensor expansion of the full field up through $O(y)$.

By evaluating the local coordinate values of the relevant bi-tensors, we obtain the local expansion of the full force in a given coordinate system (see Appendix A). As described in Ref. [6], this may be done in a systematic manner, and it is possible to obtain the explicit form of the divergence for a general orbit. However, the problem is how to decompose it into an appropriate infinite series. This is done in Section IV.

III. MODE DECOMPOSITION REGULARIZATION

We call the regularization calculation using the spherical harmonic expansion by the mode decomposition regularization. In this section, we briefly describe the regularization procedure in this approach.

The harmonic decomposition is defined by the analytic structure of the field on the two-sphere. However both the direct field and the full field have a divergence on the sphere including the particle location, the mode decomposition is ill-defined on that sphere. Therefore, we perform the harmonic decomposition of the direct and full fields on a sphere which does not include, but sufficiently close to the orbit. The steps in the mode decomposition regularization are as follows.

1) We evaluate both the full field and the direct field at

$$x = \{t, r, \theta, \phi\}\,, \tag{3.1}$$

where we do not take the coincidence limit of either t or r

2) We decompose the full force and direct force into infinte harmonic series as

$$F_{\alpha}[{}_s\phi^{\rm full}](x) = \sum_{\ell m} F^{\ell m}_{\alpha}[{}_s\phi^{\rm full}](x)\,, \tag{3.2}$$

$$F_{\alpha}[{}_s\phi^{\rm dir}](x) = \sum_{\ell m} F^{\ell m}_{\alpha}[{}_s\phi^{\rm dir}](x)\,, \tag{3.3}$$

where $F_{\alpha}[{}_s\phi^{\rm full/dir}](x)$ are expanded in terms of the spherical harmonics $Y_{\ell m}(\theta,\phi)$ with the coefficients dependent on t and r.* For the direct part, the harmonic expansion is done by extending the locally defined direct force over to the whole two-sphere in a way that correctly reproduces the divergent behavior around the orbital point z_0 up to the finite term.

3) We subtract the direct part from the full part in each ℓ, m mode to obtain

$$F^{\ell m}_{\alpha}[{}_s\phi^{\rm tail}] = (F^{\ell m}_{\alpha}[{}_s\phi^{\rm full}] - F^{\ell m}_{\alpha}[{}_s\phi^{\rm dir}])\,. \tag{3.4}$$

The we take the coincidence limit $x \to z_0$. Here we note that one can exchange the order of the procedure, i.e., first take the coincidence limit and then subtract, provided the mode coefficients are finite in the coincidence limit.

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*Rigorously speaking, the angular components of the force is expanded as $F_A = C_{\ell m}Y_{A,\ell m}$ where $Y_{A,\ell m}$ $(A=\theta,\phi)$ are the vector spherical harmonics. We also note that $F_{\alpha}[{}_s\phi_{\ell m}](x)$ and $F^{\ell m}_{\alpha}[{}_s\phi](x)$ are different since the tensorial property of the operator $F_{\alpha}[...]$ depends on the spin s of the field ${}_s\phi$.

4) Finally, by taking the sum over the modes, we obtain the self-force as

$$F_\alpha(\tau_0) = \sum_{\ell m} F_\alpha^{\ell m}[{}_s\phi^{\rm tail}](z_0)\,. \tag{3.5}$$

It should be noted that because of the divergence of the full force and direct force along a timelike orbit, the mode coefficients of the full force and the direct force are not uniquely defined when we take the coincidence limit in 3). However, the tail force is regular along the orbit [1–4], and it is uniquely defined. Therefore we expect the non-uniqueness of the direct force does not cause a problem as long as the cincidence limit is taken consistentlly for both the full force and the direct force.

IV. DECOMPOSITION OF THE DIRECT PART

The advantage of using (2.9) is that we have a systematic method for evaluating the direct part, which we describe in Subsection IV A. In Subsection IV B, we describe our method for the harmonic decomposition of the direct part.

A. Local coordinate expansion

Though we have the covariant form of the local bi-tensor expansion of the direct part, it is not useful for the derivation of the infinite series expansion of it until we evaluate it in a specific coordinate system. Here we discuss the method to evaluate the bi-tensors in a general regular coordinate system.

Before we consider the local expansion in a given coordinate system, we calculate the derivative of (2.9), and derive the direct part of the force with the local bi-tensor expansion using the equal-time condition,†

$$0 = \left[\frac{d}{d\tau}\sigma(x,z(\tau))\right]_{\tau=\tau_{\rm eq}(x)}\,. \tag{4.1}$$

We define the extension of the four-velocity off the orbit by

$$V^\alpha(x) := \bar{g}_{\alpha\bar\alpha}(x,z_{\rm eq})v_{\rm eq}^{\bar\alpha}\,, \tag{4.2}$$

where $\bar{g}_{\alpha\bar\alpha}$ is the parallel displacement bi-vector, $z_{\rm eq} = z(\tau_{\rm eq}(x))$, and $v_{\rm eq}^{\bar\alpha} = dz^{\bar\alpha}/d\tau|_{\tau=\tau_{\rm eq}(x)}$. Using the formulas in Ref. [2,3], we have

$$F_\alpha[\phi^{\rm dir}](x) = q\bar{g}_\alpha{}^{\bar\alpha}(x,z_{\rm eq})\frac{1}{\epsilon^3\kappa}\left\{\sigma_{;\bar\alpha}(x,z_{\rm eq}) + \frac{1}{3}\epsilon^2 R_{\bar\alpha\bar\beta\bar\gamma\bar\delta}(z_{\rm eq})v_{\rm eq}^{\bar\beta}\sigma^{;\bar\gamma}(x,z_{\rm eq})v_{\rm eq}^{\bar\delta}\right\} + O(y)\,, \tag{4.3}$$

$$\epsilon = \sqrt{2\sigma(x,z_{\rm eq})}\,, \tag{4.4}$$

$$\begin{aligned}\kappa &= \sqrt{-\sigma_{\bar\alpha\bar\beta}(x,z_{\rm eq})v_{\rm eq}^{\bar\alpha}v_{\rm eq}^{\bar\beta}} \\ &= 1 + \frac{1}{6}R_{\bar\alpha\bar\beta\bar\gamma\bar\delta}(z_{\rm eq})v_{\rm eq}^{\bar\alpha}\sigma^{;\bar\beta}(x,z_{\rm eq})v_{\rm eq}^{\bar\gamma}\sigma^{;\bar\delta}(x,z_{\rm eq}) + O(y^3)\,.\end{aligned} \tag{4.5}$$

The bi-tensors neccesary for the evaluation of the direct force (4.3) are $\sigma(x,\bar{x})$ and $\bar{g}_{\alpha\bar\alpha}(x,\bar{x})$, which satisfy

$$\sigma(x,\bar x) = \frac{1}{2}g^{\alpha\beta}\sigma_{;\alpha}(x,\bar x)\sigma_{;\beta}(x,\bar x) = \frac{1}{2}g^{\bar\alpha\bar\beta}\sigma_{;\bar\alpha}(x,\bar x)\sigma_{;\bar\beta}(x,\bar x)\,, \tag{4.6}$$

$$\lim_{x\to\bar x}\sigma_{;\alpha}(x,\bar x) = \lim_{x\to\bar x}\sigma_{;\bar\alpha}(x,\bar x) = 0\,, \tag{4.7}$$

$$\bar{g}_{\alpha\bar\alpha;\beta}(x,\bar x)g^{\beta\gamma}(x)\sigma_{;\gamma}(x,\bar x) = 0\,,\quad \bar{g}_{\alpha\bar\alpha;\bar\beta}(x,\bar x)g^{\bar\beta\bar\gamma}(\bar x)\sigma_{;\bar\gamma}(x,\bar x) = 0\,, \tag{4.8}$$

$$\lim_{x\to\bar x}\bar{g}_\alpha{}^{\bar\alpha} = \delta_\alpha{}^{\bar\alpha}\,. \tag{4.9}$$

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†Actually, the equal-time condition is not essential for the evaluation of the direct part. However, one can see the dependence on the spin of the field in a transparent manner under the equal-time condition as we describe in Appendix B.

In addition, we need the generalized van Vleck-Morette determinant,

$$\Delta(x,\bar{x}) = \det\bigl(-\bar{g}^{\alpha\bar{\alpha}}(x,\bar{x})\sigma_{;\bar{\alpha}\beta}(x,\bar{x})\bigr) . \tag{4.10}$$

We consider the local coordinate expansion of these bi-tensors around the coincidence limit $x \to \bar{x}$, assuming that we have no coordinate singularity at $\bar{x}$.

In the coincidence limit, the effect of the curvature is small and we know the exact forms for half the geodesic distance bi-scalar and the paralell displacement bi-vector in the locally Cartesian coordinates as

$$\begin{aligned} \sigma(x,\bar{x}) &= \frac{1}{2}\eta_{\alpha\beta}(x^\alpha - \bar{x}^\alpha)(x^\beta - \bar{x}^\beta) + O(|x-\bar{x}|^3) , \\ \bar{g}_{\alpha\bar{\alpha}}(x,\bar{x}) &= \eta_{\alpha\bar{\alpha}} + O(|x-\bar{x}|) . \end{aligned} \tag{4.11}$$

Therefore, in a general regular coordinate system, $\sigma(x,\bar{x})$ and $\bar{g}_{\alpha\bar{\alpha}}(x,\bar{x})$ can be expanded as

$$\sigma(x,\bar{x}) = \frac{1}{2}g_{\alpha\beta}(\bar{x})y^{\alpha\beta} + \sum_{n=3,4,\cdots}\frac{1}{n!}A_{\alpha^1\alpha^2\cdots\alpha^n}(\bar{x})y^{\alpha^1\alpha^2\cdots\alpha^n} , \tag{4.12}$$

$$\bar{g}_{\alpha\bar{\alpha}}(x,\bar{x}) = g_{\alpha\bar{\alpha}}(\bar{x}) + \sum_{n=1,2,\cdots}\frac{1}{n!}B_{\alpha\bar{\alpha}|\beta^1\beta^2\cdots\beta^n}(\bar{x})y^{\beta^1\beta^2\cdots\beta^n} , \tag{4.13}$$

where

$$y^{\alpha^1\alpha^2\cdots} = (x^{\alpha^1} - \bar{x}^{\alpha^1})(x^{\alpha^2} - \bar{x}^{\alpha^2})\cdots . \tag{4.14}$$

To calculate the reaction force to a monopole particle, it is enough to know the expansion coefficients of $n = 3,4$ of Eq. (4.12) and $n = 1,2$ of Eq. (4.13).[‡] For a general metric, from (4.6) and (4.8), we have

$$\begin{aligned} A_{\alpha\beta\gamma} &= \frac{3}{2}g_{(\alpha\beta,\gamma)} , &(4.15)\\ A_{\alpha\beta\gamma\delta} &= 2g_{(\alpha\beta,\gamma\delta)} - g_{\mu\nu}\Gamma^\mu_{(\alpha\beta}\Gamma^\nu_{\gamma\delta)} , &(4.16)\\ B_{\alpha\beta|\gamma} &= \Gamma_{\beta|\alpha\gamma} , &(4.17)\\ B_{\alpha\beta|\gamma\delta} &= \frac{1}{2}\left(\Gamma_{\beta|\alpha\gamma,\delta} + \Gamma_{\beta|\alpha\delta,\gamma} - g_{\mu\nu}\Gamma^\mu_{\alpha\gamma}\Gamma^\nu_{\beta\delta} - g_{\mu\nu}\Gamma^\mu_{\alpha\delta}\Gamma^\nu_{\beta\gamma}\right) . &(4.18) \end{aligned}$$

The explicit evaluation of these coefficients in the Boyer-Lindquist coordinates is given in Appendix A.

The local expansion of the force (4.3) on the Boyer-Lindquist coordinates is quite tedious, though systematic. We implement this calculation using *Maple(R)*, an algebraic calculation program, and have 100 pages output in the end. However, most of the terms make a vanishing contribution to the harmonic coefficients in the coincidence limit $x \to z_0$. Below, we shall focus on the terms that are non-vanishing in the coincidence limit.

B. Harmonic decomposition of the direct part

Without loss of generality, we may assume that the particle is located at $\theta_0 = \pi/2$, $\phi_0 = 0$ at time t_0. Since the full force is calculated in the form of the Fourier-harmonic expansion and the Fourier modes are independent of the spherical harmonics, we may take the field point to lie on the hypersurface $t = t_0$ in the full force. Hence we may take $t = t_0$ before we perform the local coordinate expansion of the direct force. That is, we consider the local coordinate expansion of the direct force at a point $\{t_0, r, \theta, \phi\}$ near the particle localtion $\{t_0, r_0, \pi/2, 0\}$.

The local expansion of the direct force on the Boyer-Lindquist coordinates can be done in such a way that it consists of terms of the form,

$$\frac{R^{n_1}\Theta^{n_2}\phi^{n_3}}{\xi^{2n_4+1}} , \tag{4.19}$$

where n_1, n_2, n_3, n_4 are non-negative integers, and

[‡] In the calculation of the direct force given below, only Eq. (4.12) but Eq. (4.13) turns out to be necessary. This is a result of our choice of the off-world line extension of the direct force, i.e., the parallel-propagation extension (4.2).

$$\xi := \sqrt{2}r_0\left(a-\cos\tilde{\theta}+\frac{b}{2}(\phi-\phi')^2\right)^{1/2}, \tag{4.20}$$

$$R := r-r_0\,,\quad \Theta := \theta-\frac{\pi}{2}\,, \tag{4.21}$$

with a, b and ϕ' defined by

$$a := 1+\frac{1}{2r_0^2}\frac{r_0^2}{r_0^2+\mathcal{L}^2}\frac{r_0^2}{(r_0-2M)^2}\mathcal{E}^2R^2\,, \tag{4.22}$$

$$b := \frac{\mathcal{L}^2}{r_0^2}\,, \tag{4.23}$$

$$\phi' := -\frac{\mathcal{L}}{r_0^2+\mathcal{L}^2}u_rR\,, \tag{4.24}$$

where $\mathcal{E} := -g_{tt}dt/d\tau$, $\mathcal{L} := g_{\phi\phi}d\phi/d\tau$, and $u_r := g_{rr}dr/d\tau$, and $\tilde{\theta}$ is the relative angle between (θ,ϕ) and $(\pi/2,\phi')$.

There are two apparently different terms in the covariant form of the direct force given by Eq. (4.3); the first term in the curly brackets exhibiting the quadratic divergence, and the second term proportional to the curvature tensor that appears to be finite in the coincidence limit. In the local coordinate expansion, the second term will give terms of the form R/ξ or ϕ/ξ. As shown in Section V, the harmonic coefficients of R/ξ vanish in the coincidence limit, while those of ϕ/ξ are finite but they give no contribution to the final result when the infinite harmonic modes are summed up after the coincidence limit is taken. Hence we may focus on the first term. In passing, it is worthwhile to note the following fact. Since the direct force shows its dependence on the spin of the field only through this second term (see Appendix B), the harmonic coefficients of the direct force, which are to be subtracted from the full force, will be independent of the spin of the field. [§]

Let us focus on the first term in the curly brackets of Eq. (4.3). Since the orbit always remains on the equatorial plane, the force is symmetric under the transformation $\theta\to\pi-\theta$, which implies there is no term proportional to odd powers of Θ. Hence we only need to consider the case of n_2 being an even number in the general form given by Eq. (4.19). Then the factor Θ^{n_2} may be eliminated by expressing Θ^2 in terms of ξ, R and ϕ, and we are left with terms of the form,

$$\frac{R^{n_1}\phi^{n_3}}{\xi^{2n_4+1}}\,. \tag{4.25}$$

Explicitly, we find

$$\begin{aligned}F_t^{\rm dir} = q\Bigg(&\mathcal{E}u_r\frac{R}{\xi^3}+\mathcal{E}\mathcal{L}\frac{\phi}{\xi^3}\\
&-\frac{1}{2}\frac{(r_0-2M)\mathcal{E}u_r}{r_0^2}\frac{1}{\xi}+\frac{2(r_0-2M)\mathcal{E}\mathcal{L}^2u_r}{r_0^2}\frac{\phi^2}{\xi^3}\\
&-\frac{3}{2}\frac{(r_0-2M)\mathcal{E}\mathcal{L}^4u_r}{r_0^2}\frac{\phi^4}{\xi^5}\Bigg)\,,\end{aligned} \tag{4.26}$$

$$\begin{aligned}F_r^{\rm dir} = q\Bigg(&\frac{\mathcal{L}^2}{r_0(r_0-2M)}\frac{R}{\xi^3}-\frac{r_0^2\mathcal{E}^2}{(r_0-2M)^2}\frac{R}{\xi^3}-\mathcal{L}u_r\frac{\phi}{\xi^3}\\
&-\frac{1}{2}\frac{2r_0^2+\mathcal{L}^2}{r_0^3}\frac{1}{\xi}+\frac{1}{2}\frac{\mathcal{E}^2}{r_0-2M}\frac{1}{\xi}\\
&+\frac{1}{2}\frac{(3r_0^2+4\mathcal{L}^2)\mathcal{L}^2}{r_0^3}\frac{\phi^2}{\xi^3}-\frac{2\mathcal{E}^2\mathcal{L}^2}{r_0-2M}\frac{\phi^2}{\xi^3}\\
&-\frac{3}{2}\frac{(r_0^2+\mathcal{L}^2)\mathcal{L}^4}{r_0^3}\frac{\phi^4}{\xi^5}+\frac{3}{2}\frac{\mathcal{E}^2\mathcal{L}^4}{r_0-2M}\frac{\phi^4}{\xi^5}\,,\end{aligned} \tag{4.27}$$

[§]This is also a result of the specific off-worldline extension chosen for the four-velocity. It is valid for the parallel-propagation extension, but does not hold, in general, for other extensions.

$$F_\theta^{\rm dir} = 0\,, \tag{4.28}$$

$$F_\phi^{\rm dir} = q\Bigg(-\mathcal{L}u_r\frac{R}{\xi^3} - (r_0^2+\mathcal{L}^2)\frac{\phi}{\xi^3} + \frac{1}{2}\frac{(r_0-2M)\mathcal{L}u_r}{r_0^2}\frac{1}{\xi} - \frac{1}{2}\frac{(r_0-2M)(r_0^2+4\mathcal{L}^2)\mathcal{L}u_r}{r_0^2}\frac{\phi^2}{\xi^3} + \frac{3}{2}\frac{(r_0-2M)(r_0^2+\mathcal{L}^2)\mathcal{L}^3u_r}{r_0^2}\frac{\phi^4}{\xi^5}\Bigg)\,, \tag{4.29}$$

where $F_\alpha^{\rm dir} = F_\alpha[\phi^{\rm dir}]$. The absence of $F_\theta^{\rm dir}$ is because of the symmetry of the background; the orbit remains on the equatorial plane even under the action of the self-force. In the above, we have discarded the terms of the form R/ξ or ϕ/ξ. As mentioned before, and as shown in Section V, such terms give no contribution to the final force.

What we have to do now is to perform the harmonic decomposition of the components of the direct force given above. To do so, we note the following important fact. Apart from the trivial multiplicative factor of R^{n_1} which is independent of the spherical coordinates, the terms to be expanded in the spherical harmonics are of the form ϕ^{n_3}/ξ^{2n_4+1}, or $(\phi-\phi')^{n_3}/\xi^{2n_4+1}$. To the order of accuracy we need (in fact only the leading order accuracy is sufficient; see Appendix C), the factor $(\phi-\phi')^{n_3}$ may be eliminated by replacing it to an equivalent ϕ-derivative operator of degree n_3 acting on $\xi^{2n_3-2n_4-1}$, which is further converted to a polynomial in m after the harmonic expansion of $\xi^{2n_3-2n_4-1}$. Thus the only basic formula we need is the harmonic expansion of ξ^{2p-1} where p is an integer. Detailed derivation of it is given in Appendix D. Note that, apart from the term $b(\phi-\phi')^2/2$ in ξ with respect to which we expand ξ in a convergent infinite series, ξ^{2p-1} is defined over the whole sphere to allow the straightforward harmonic decomposition. The result to the leading order in the coincidence limit $a\to 1+0$ is

$$\left(\frac{\xi}{\sqrt{2}\,r_0}\right)^{2p-1} = \left(a-\cos\tilde\theta+\frac{b}{2}(\phi-\phi')^2\right)^{p-1/2} = 2\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m}^{p-1/2}(a)Y_{\ell m}(\theta,\phi)Y_{\ell m}^*(\theta',\phi')\,, \tag{4.30}$$

$$D_{\ell m}^{p-1/2}(a) \to \begin{cases} \dfrac{1}{\sqrt{1+b}}\dfrac{1}{-p-1/2}(a-1)^{p+1/2}\,, & \text{for } p+\frac{1}{2}<0,\\[2ex] \dfrac{(-1)^\ell 2^{p+1/2}}{\sqrt{1+b}}\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma(p+1/2)\Gamma(p+n+1/2)}{\Gamma(p+n-\ell+1/2)\Gamma(p+n+\ell+3/2)}\frac{1}{n!}\left(\frac{-m^2b}{1+b}\right)^n\,, & \text{for } p+\frac{1}{2}>0\,. \end{cases} \tag{4.31}$$

We note that, although what we need here is only the case of an integer p, the above formula is valid for any p (except for the case $p=-1/2$).

After the decomposition, we can take the radial coincidence limit $r\to r_0$ (followed by the angular coincidence limit if desired). The basic properties of the resulting mode coefficients in the coincidence limit are discussed in Section V. Here we briefly explain the reason why the terms proportional to R/ξ and ϕ/ξ give no contribution to the final result. The term R/ξ corresponds to R times the case of $p=0$, for which $D_{\ell m}^{p-1/2}$ is finite in the limit $a\to 1$ (i.e., $R\to 0$). Hence all the coefficients vanish in the radial coincidence limit. As for ϕ/ξ, it can be replaced by $(\phi-\phi')/\xi$ which is equivalent to $\partial_\phi\xi$ in the coincidence limit. This corresponds to the case of $p=1$ multiplied by m. Hence all the harmonic coefficients become odd functions of m, and their sum over m for each ℓ vanishes in the angular coincidence limit. As a result, the non-vanishing contribution comes only from the terms R/ξ^3, ϕ/ξ^3, $1/\xi$, ϕ^2/ξ^3 and ϕ^4/ξ^5.

V. REGULARIZATION COUNTER TERMS

In this section, we present the mode decomposition of the direct force given by Eqs. (4.26) $\sim$ (4.29), and compare the resulting regularization counter terms with those obtained by Barack and Ori [16] in their mode-sum regularization scheme (MSRS) [9].

Barack and Ori define the regularization counter terms as

$$\lim_{x\to z_0} F_{\alpha l}^{\rm dir} = A_\alpha L + B_\alpha + C_\alpha/L + O(L^{-2})\,. \tag{5.1}$$

$$D_\alpha = \sum_{l=0}^{\infty}\left[\lim_{x\to z_0} F_{\alpha l}^{\rm dir} - A_\alpha L - B_\alpha - C_\alpha/L\right]\,. \tag{5.2}$$

where $F_{\alpha l}^{\rm dir}$ is the multipole l-mode of $F_\alpha^{\rm dir}$, $L=\ell+1/2$, and A_α, B_α and C_α are independent of L. The A_α term is to subract the quadratic divergence, the B_α term the linear divergence, and the C_α term the logarithmic divergence.

The D_α term is the remaining finite contribution of the direct force to be subtracted. As shown in Appendix C, we find $C_\alpha = D_\alpha = 0$ in agreement with Barack and Ori [9]. We also find the complete agreement of A_α and B_α terms with their results for a general geodesic orbit [15,16] as given below.

The direct part of the force to be considered has the form given by Eq. (4.25), which may be re-written as

$$\frac{R^{n_1}(\phi-\phi')^{n_3}}{\xi^{2n_4+1}}\,, \tag{5.3}$$

where (n_1, n_3, n_4) are non-negative integers. Because the highest order of divergence is quadratic, it is sufficient to consider the cases $n_1 + n_3 - 2n_4 = -1$, 0 and 1.**

We first note that

$$\frac{\phi-\phi'}{\xi^{2n_4+1}} = -\frac{1}{2n_4-1}\frac{1}{r_0^2+\mathcal{L}^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\xi^{2n_4-1}}\right) + O(y^{-2n_4+2})\,. \tag{5.4}$$

By using this equation recursively, we obtain

$$\frac{(\phi-\phi')^{n_3}}{\xi^{2n_4+1}} \propto \frac{\partial^{n_3}}{\partial\phi^{n_3}}\xi^{2n_3-2n_4-1} + O(y^{n_3-2n_4+1})\,.$$

In the context of the harmonic decomposition, we may replace the derivative $\partial/\partial\phi$ by $i\,m$. Hence instead of Eq. (5.3), we may consider the terms of the form,

$$m^{n_3}R^{n_1}\xi^{2n_3-2n_4-1} + O(y^{n_1+n_3-2n_4+1})\,. \tag{5.5}$$

In the above equation, we have indicated by $O(y^{n_1+n_3-2n_4+1})$ the presence of correction terms of $O(y^2)$ relative to the original form (5.3). In terms of the regularization parameters A_α, B_α, C_α and D_α, this implies the terms with $n_1+n_3-2n_4 = -1$ would contribute to A_α and C_α, and the terms with $n_1+n_3-2n_4 = 0$ to B_α and D_α, while the terms with $n_1+n_3-2n_4 = 1$ to D_α. However, as shown in Appendix C, by a general argument, we can show that both C_α and D_α vanish. Therefore we do not have to worry about the $O(y^2)$ corrections in Eq. (5.5) but may focus on the leading behavior of it in the coincidence limit.

Keeping this fact in mind, we now analyze which cases of the form (5.5) contribute to the regularization parameters A_α and B_α. For this purpose, we set $n_1+n_3-2n_4 = q$ where $q = -1$, 0 or 1. Then comparing Eq. (5.5) with Eq. (4.30), we find it is convenient to separately consider the two cases:

(1) The case $2p = 2n_3 - 2n_4 = n_3 - n_1 + q \le -2$.
In this case, the harmonic coefficients of Eq. (5.5) behave as

$$\sim R^{n_1+2n_3-2n_4+1} = R^{n_3+q+1}\,.$$

Since $n_3 \ge 0$, the harmonic coefficients are non-vanishing in the limit $R\to 0$ only if $n_3 = 0$ and $q = -1$. This means $n_1 = 2n_4 - 1$ (≥ 0). Therefore only the terms of the form R^{2n_4-1}/ξ^{2n_4+1} $(n_4 \ge 1)$ give finite coefficients, and they contribute to A_α.

(2) The case $2p = 2n_3 - 2n_4 = n_3 - n_1 + q \ge 0$.
In this case, since the harmonic coefficients of $\xi^{2n_3-2n_4-1}$ are finite in the limit $R \to 0$, we must have $n_1 = 0$, hence $n_3 = 2n_4 + q$. Therefore, since $D_{\ell m}^{p-1/2}$ is an even function of m, the harmonic coefficents will be odd functions of m if q is odd, i.e., if $q = -1$ or 1. When the sum over m is taken, the result vanishes in the angular coincidence limit if q is odd because of the symmetry property of $|Y_{\ell m}(\theta,\phi)|^2$ under $m \to -m$. Thus only the case of $q = 0$ or $n_3 = 2n_4$ remains. The corresponding terms are of the form $(\phi-\phi')^{2n_4}/\xi^{2n_4+1}$, and they contribute to B_α.

From the above results, and noting that $\phi' \propto R$, we obtain the equality,

$$\frac{(\phi-\phi')^{2n}}{\xi^{2n+1}} = \frac{(\phi^2-2\phi\phi'+\phi'^2)^n}{\xi^{2n+1}} = \frac{\phi^2}{\xi^{2n+1}}\,, \tag{5.6}$$

which holds in the sense of its contributions to the regularization parameters. Thus, to summarize, the non-vanishing contributions are from the terms either of the form R^{2n+1}/ξ^{2n+3} or of the form ϕ^{2n}/ξ^{2n+1}, where n is a non-negative integer, and the former contributes to A_α while the latter to B_α.

**Although we may further restrict n_4 to the range $0 \le n_4 \le 2$ from the explict form of the direct force in Eqs. (4.26) $\sim$ (4.29), we choose not to do so because it turns out to be unnecessary in the following discussion.

A. The A-term

The A-term describes the quadratic divergent terms of the direct force. Thus we consider the most divergent terms in Eqs. (4.26) $\sim$ (4.29),

$$\frac{R}{\xi^3} \quad \text{and} \quad \frac{\phi}{\xi^3}\,. \tag{5.7}$$

As discussed above, the term ϕ/ξ^3 may be replaced as $(\phi-\phi')/\xi+\phi'/\xi=\phi'/\xi$. Hence we may focus on the form R/ξ^3. The essential fact is that this is odd in R. This leads to the harmonic coefficients proportional to sign(R). Using the fomula (D4), we obtain

$$A_t = \mathrm{sign}(R)\frac{q^2}{r^2}\frac{r_0-2M}{r_0}\frac{u_r}{1-\mathcal{L}^2/r_0^2}\,, \tag{5.8}$$

$$A_r = -\mathrm{sign}(R)\frac{q^2}{r^2}\frac{r_0}{r_0-2M}\frac{\mathcal{E}}{1-\mathcal{L}^2/r_0^2}\,, \tag{5.9}$$

$$A_\phi = 0\,. \tag{5.10}$$

These A-terms vanish when averaged over both limits $R\to\pm 0$. There could be correction terms of $O(y^0)$ which could contribute to the C and D-terms. However, as shown in Appendix C, they are known to be absent.

B. The B-term

The linearly divergent terms are described by the B-term, which are of the form,

$$\frac{\phi^{2n}}{\xi^{2n+1}}\,, \tag{5.11}$$

in Eqs. (4.26) $\sim$ (4.29). The Legendre coefficients are given by the formula (D15). We find

$$B_t = -\frac{(r_0-2M)\mathcal{E}u_r}{2r_0}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}(-1)^n\frac{(2n+1)^2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\frac{\mathcal{L}^{2n}}{r_0^{2n+1}}\,, \tag{5.12}$$

$$\begin{aligned} B_r = {} & \frac{(r_0-2M)u_r^2}{2r_0}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}(-1)^n\frac{(2n+1)^2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\frac{\mathcal{L}^{2n}}{r_0^{2n+1}} \\ & -\frac{1}{2r_0}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}(-1)^n\frac{(-(2n-1))\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\frac{\mathcal{L}^{2n}}{r_0^{2n+1}}\,, \end{aligned} \tag{5.13}$$

$$\begin{aligned} B_\phi = {} & \frac{(r_0-2M)\mathcal{L}u_r}{2r_0^2}\Bigg(\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}(-1)^n\frac{(2n+1)^2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\frac{\mathcal{L}^{2n}}{r_0^{2n+1}} \\ & -\frac{r_0^2}{\mathcal{L}^2}\sum_{n=0}^{\infty}\frac{(2(n+1))!}{2^{2(n+1)}((n+1)!)^2}(-1)^n\frac{(2n+1)^2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\frac{\mathcal{L}^{2(n+1)}}{r_0^{2(n+1)+1}}\Bigg)\,. \end{aligned} \tag{5.14}$$

The above may be expressed in terms of the hypergeometric functions as

$$B_t = -\frac{(r_0-2M)\mathcal{E}u_r}{2r_0^3}F\left(\frac{3}{2},\frac{3}{2};1;-\frac{\mathcal{L}^2}{r_0^2}\right)\,, \tag{5.15}$$

$$B_r = \frac{(r_0-2M)u_r^2}{2r_0^3}F\left(\frac{3}{2},\frac{3}{2};1;-\frac{\mathcal{L}^2}{r_0^2}\right)-\frac{1}{2r_0^2}\left(F\left(\frac{1}{2},\frac{1}{2};1;-\frac{\mathcal{L}^2}{r_0^2}\right)+\frac{\mathcal{L}^2}{2r_0^2}F\left(\frac{3}{2},\frac{3}{2};2;-\frac{\mathcal{L}^2}{r_0^2}\right)\right)\,, \tag{5.16}$$

$$B_\phi = \frac{(r_0-2M)\mathcal{L}u_r}{16r_0^3}\left(8F\left(\frac{3}{2},\frac{3}{2};1;-\frac{\mathcal{L}^2}{r_0^2}\right)-4F\left(\frac{3}{2},\frac{3}{2};2;-\frac{\mathcal{L}^2}{r_0^2}\right)+\frac{9\mathcal{L}^2}{r_0^2}F\left(\frac{5}{2},\frac{5}{2};3;-\frac{\mathcal{L}^2}{r_0^2}\right)\right)\,. \tag{5.17}$$

The above results for the A and B-terms perfectly agree with the results obtained by Barack and Ori in a quite different fashion [15,16]. [††]

[††]Here we give the values of B_α expressed in terms of generalized hypergeometric functions, while in [15] these are given in

VI. CONCLUSION AND DISCUSSION

Our final goal is to establish a method of calculation of the local gravitational reaction force to a point particle orbiting a Kerr black hole. We have pointed out in Section I that there are two problems; the 'subtraction problem' and the 'gauge problem'.

In this paper, we have only discussed a possible approach to the subtraction problem. We have introduced a regularization method which utilizes the spherical-harmonic decomposition, and have derived the direct part of the self-force, which turns out to be independent of the spin s of the field under consideration. The harmonic decomposition of this direct part has been carried out, and the regularization counter terms for the self-force have been derived for a general geodesic orbit. We have found our result agrees completely with the result obtained by Barack and Ori [16] in their mode-sum regularization scheme (MSRS) [9].

To compare with the MSRS, we have derived the regularization counter terms which are obtained by summating the harmonic coefficients over m. However, when we extend our method to the Kerr background, we may have to carry out the regularization before taking the m-summation. In this sense, the formulas derived in Appendix D, where no summation over m is assumed, may be still useful in the Kerr case.

It is worthwhile to point out that the gauge problem in the gravitational case seems far more serious than the subtraction problem. What we know at the moment is that the gravitational self-force is described by the tail part of the metric perturbation induced by a particle [3,4]. However this is justified only in the harmonic gauge, while the full metric perturbation can be obtained only in the Regge-Wheeler gauge or in the radiation gauge where the identification of the tail part is highly non-trivial. A prescription to identify the tail part of the metric perturbation has been proposed in [10], but it needs to be verified. The gauge problem for the non-radiative monopole and dipole components of the metric perturbation which are not obtainable in the Teukolsky formalism seems to stand as additional serious obstacle. Possible resolutions for the gauge problem are under investigation. ‡‡

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APPENDIX A: BI-TENSORS AND LOCAL COORDINATE EXPANSION

Bi-tensors are tensors which depend on two distinct field points, say, x^α and $\bar{x}^{\bar\alpha}$. For our purpose, we consider half the squared geodesic interval bi-scalar $\sigma(x,\bar{x})$, and the geodesic paralell displacement bi-vector $\bar{g}_{\alpha\bar\alpha}(x,\bar{x})$, which satisfy

$$\sigma(x,\bar{x}) \;=\; \frac{1}{2}g^{\alpha\beta}\sigma_{;\alpha}(x,\bar{x})\sigma_{;\beta}(x,\bar{x}) \;=\; \frac{1}{2}g^{\bar\alpha\bar\beta}\sigma_{;\bar\alpha}(x,\bar{x})\sigma_{;\bar\beta}(x,\bar{x})\,, \tag{A1}$$

$$\lim_{x\to\bar{x}}\sigma_{;\alpha}(x,\bar{x}) \;=\; \lim_{x\to\bar{x}}\sigma_{;\bar\alpha}(x,\bar{x}) \;=\; 0\,, \tag{A2}$$

$$\bar{g}_{\alpha\bar\alpha;\beta}(x,\bar{x})g^{\beta\gamma}(x)\sigma_{;\gamma}(x,\bar{x}) \;=\; 0\,, \quad \bar{g}_{\alpha\bar\alpha;\bar\beta}(x,\bar{x})g^{\bar\beta\bar\gamma}(\bar{x})\sigma_{;\bar\gamma}(x,\bar{x}) \;=\; 0\,, \tag{A3}$$

$$\lim_{x\to\bar{x}}\bar{g}_\alpha{}^{\bar\alpha} \;=\; \delta_\alpha{}^{\bar\alpha}\,. \tag{A4}$$

We also need the generalized van Vleck-Morette determinant bi-scalar,

$$\Delta(x,\bar{x}) = \det\left(-\bar{g}^{\alpha\bar\alpha}(x,\bar{x})\sigma_{;\bar\alpha\beta}(x,\bar{x})\right). \tag{A5}$$

We consider the local expansion of these bi-tensors around the coincidence limit $x\to\bar{x}$.

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terms of the two complete elliptic integrals K and E. The two expressions are related by changing of variables and using the formulas in [18].

‡‡In this respect, recently some progress has been made in [17].

In calculating the local expansion of the field and its derivatives in a covariant way, the following formula are useful [3].

$$\sigma_{;\alpha\beta}(x,z) = g_{\alpha\beta}(z) - \frac{1}{3}R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}(z)\sigma_{;\gamma}(x,z)\sigma_{;\delta}(x,z) + O(\epsilon^3)\,, \tag{A6}$$

$$\begin{aligned}\sigma_{;\mu\beta}(x,z) &= -\bar{g}_{\mu}{}^{\alpha}(x,z)\left(g_{\alpha\beta}(z) + \frac{1}{6}R_{\alpha\gamma\beta\delta}(z)\sigma^{;\gamma}(x,z)\sigma^{;\delta}(x,z)\right)\\ &\quad +O(\epsilon^3)\,,\end{aligned} \tag{A7}$$

$$\begin{aligned}\bar{g}^{\mu\alpha}{}_{;\beta}(x,z) &= -\frac{1}{2}\bar{g}^{\mu\gamma}(x,z)R^{\alpha}{}_{\gamma\beta\delta}(z)\sigma^{;\delta}(x,z) + O(\epsilon^2)\,,\\ \bar{g}^{\mu\alpha}{}_{;\nu}(x,z) &= -\frac{1}{2}\bar{g}^{\mu\beta}(x,z)\bar{g}_{\nu}{}^{\gamma}(x,z)R^{\alpha}{}_{\beta\gamma\delta}(z)\sigma^{;\delta}(x,z) + O(\epsilon^2)\,.\end{aligned} \tag{A8}$$

As for the calculation of $\Delta(x,\bar{x})$, we use the result of the covariant expansion given in Ref. [2]. We have

$$\sigma_{;\bar{\alpha}\beta}(x,\bar{x}) = -\bar{g}_{\beta}{}^{\bar{\beta}}(x,\bar{x})\left(g_{\bar{\alpha}\bar{\beta}}(\bar{x}) + \frac{1}{6}R_{\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\delta}}(\bar{x})\sigma^{;\bar{\gamma}}(x,\bar{x})\sigma^{;\bar{\delta}}(x,\bar{x}) + O(|x-\bar{x}|^3)\right). \tag{A9}$$

Then, for a vacuum background, we have

$$\Delta(x,\bar{x}) = 1 + O(|x-\bar{x}|^3)\,. \tag{A10}$$

The local expansion of these bi-tensors by the background coordinates is derived from the formulas (4.12) and (4.13), and the expansion coefficiets expressed in terms of the ordinary derivatives of the background metric are given in Eqs. (4.15), (4.16), (4.17) and (4.18). In the Schwarzschild background, the non-vanishing components of these coefficients are

$$\begin{aligned}
&A_{ttr} = A_{trt} = A_{rtt} = -\frac{M}{r^2}\,,\\
&A_{rrr} = -\frac{3M}{(r-2M)^2}\,,\\
&A_{r\theta\theta} = A_{\theta r\theta} = A_{\theta\theta r} = r\,,\\
&A_{r\phi\phi} = A_{\phi r\phi} = A_{\phi\phi r} = r\sin^2\theta\,,\\
&A_{\theta\phi\phi} = A_{\phi\phi\theta} = A_{\phi\theta\phi} = r^2\sin\theta\cos\theta\,,\\
&A_{tttt} = -\frac{M^2(r-2M)}{r^5}\,,\\
&A_{ttrr} = A_{trtr} = A_{rttr} = A_{trrt} = A_{rtrt} = A_{rrtt} = \frac{M(4r-5M)}{3r^3(r-2M)}\,,\\
&A_{tt\theta\theta} = A_{t\theta t\theta} = A_{\theta tt\theta} = A_{t\theta\theta t} = A_{\theta t\theta t} = A_{\theta\theta tt} = \frac{M(r-2M)}{3r^2}\,,\\
&A_{tt\phi\phi} = A_{t\phi t\phi} = A_{\phi tt\phi} = A_{t\phi\phi t} = A_{\phi t\phi t} = A_{\phi\phi tt} = \frac{M(r-2M)}{3r^2}\sin^2\theta\,,\\
&A_{rrrr} = \frac{M(8r-M)}{r(r-2M)^3}\,,\\
&A_{rr\theta\theta} = A_{r\theta r\theta} = A_{\theta rr\theta} = A_{r\theta\theta r} = A_{\theta r\theta r} = A_{\theta\theta rr} = -\frac{M}{3(r-2M)}\,,\\
&A_{rr\phi\phi} = A_{r\phi r\phi} = A_{\phi rr\phi} = A_{r\phi\phi r} = A_{\phi r\phi r} = A_{\phi\phi rr} = -\frac{M}{3(r-2M)}\sin^2\theta\,,\\
&A_{\theta\theta\theta\theta} = -r(r-2M)\,,\\
&A_{\theta\theta\phi\phi} = A_{\theta\phi\theta\phi} = A_{\phi\theta\theta\phi} = A_{\theta\phi\phi\theta} = A_{\phi\theta\phi\theta} = A_{\phi\phi\theta\theta} = -\frac{r(3r-2M)}{3}\sin^2\theta\,,\\
&A_{\phi\phi\phi\phi} = -r(r-2M)\sin^4\theta - r^2\sin^2\theta\cos^2\theta\,,\\
&A_{r\theta\phi\phi} = A_{r\phi\theta\phi} = A_{r\phi\phi\theta} = A_{\theta r\phi\phi} = A_{\phi r\theta\phi} = A_{\phi r\phi\theta} = A_{\theta\phi r\phi}\\
&\quad = A_{\phi\theta r\phi} = A_{\phi\phi r\theta} = A_{\theta\phi\phi r} = A_{\phi\theta\phi r} = A_{\phi\phi\theta r} = r\sin\theta\cos\theta\,,\\
&B_{tt|r} = -B_{tr|t} = B_{rt|t} = -\frac{M}{r^2}\,,
\end{aligned}$$

$$
\begin{aligned}
&B_{rr|r} = -\frac{M}{(r-2M)^2}\,,\\
&B_{\theta\theta|r} = -B_{\theta r|\theta} = B_{r\theta|\theta} = r\,,\\
&B_{\phi\phi|r} = -B_{\phi r|\phi} = B_{r\phi|\phi} = r\sin^2\theta\,,\\
&B_{\theta\phi|\phi} = -B_{\phi\theta|\phi} = B_{\phi\phi|\theta} = r^2\sin\theta\cos\theta\,,\\
&B_{tt|tt} = -\frac{M^2(r-2M)}{r^5}\,,\\
&B_{tt|rr} = \frac{M(2r-3M)}{r^3(r-2M)}\,,\\
&B_{tr|tr} = B_{tr|rt} = -\frac{M(r-3M)}{r^3(r-2M)}\,,\\
&B_{rt|tr} = B_{rt|rt} = \frac{M(r-M)}{r^3(r-2M)}\,,\\
&B_{rr|tt} = \frac{M^2}{r^3(r-2M)}\,,\\
&B_{t\theta|t\theta} = B_{t\theta|\theta t} = B_{\theta t|t\theta} = B_{\theta t|\theta t} = \frac{M(r-2M)}{2r^2}\,,\\
&B_{t\phi|t\phi} = B_{t\phi|\phi t} = B_{\phi t|t\phi} = B_{\phi t|\phi t} = \frac{M(r-2M)}{2r^2}\sin^2\theta\,,\\
&B_{rr|rr} = \frac{M(2r-M)}{r(r-2M)^3}\,,\\
&B_{rr|\theta\theta} = -1\,,\\
&B_{r\theta|r\theta} = B_{r\theta|\theta r} = -\frac{M}{2(r-2M)}\,,\\
&B_{\theta r|r\theta} = B_{\theta r|\theta r} = -\frac{2r-3M}{2(r-2M)}\,,\\
&B_{rr|\phi\phi} = -\sin^2\theta\,,\\
&B_{r\phi|r\phi} = B_{r\phi|\phi r} = -\frac{M}{2(r-2M)}\sin^2\theta\,,\\
&B_{\phi r|r\phi} = B_{\phi r|\phi r} = -\frac{2r-3M}{2(r-2M)}\sin^2\theta\,,\\
&B_{\theta\theta|\theta\theta} = -r(r-2M)\,,\\
&B_{\theta\theta|\phi\phi} = -r^2\cos^2\theta\,,\\
&B_{\theta\phi|\theta\phi} = B_{\theta\phi|\phi\theta} = -r(r-M)\sin^2\theta\,,\\
&B_{\phi\theta|\theta\phi} = B_{\phi\theta|\phi\theta} = -r^2\cos^2\theta + Mr\sin^2\theta\,,\\
&B_{\phi\phi|\theta\theta} = -r^2\sin^2\theta\,,\\
&B_{\phi\phi|\phi\phi} = -r(r-2M)\sin^4\theta - r^2\sin^2\theta\cos^2\theta\,,\\
&B_{r\theta|\phi\phi} = B_{\theta r|\phi\phi} = -B_{\theta\phi|r\phi} = -B_{\theta\phi|\phi r} = B_{\phi\theta|r\phi} = B_{\phi\theta|\phi r} = -B_{r\phi|\theta\phi}\\
&\quad = -B_{r\phi|\phi\theta} = B_{\phi r|\theta\phi} = B_{\phi r|\phi\theta} = -B_{\phi\phi|r\theta} = -B_{\phi\phi|\theta r} = -r\sin\theta\cos\theta\,.
\end{aligned}
$$

APPENDIX B: THE DIRECT PART OF ELECTROMAGNETIC AND GRAVITATIONAL SELF-FORCE

In this appendix, we summarize the direct part of the vector and tensor field. The direct part of the field is obtained by integrating the direct part of the Green function ${}_sG^{\rm dir}_{\{A\}}$ as same as the scalar case.

$$
{}_sG^{\rm dir}_{\{A\}}(x,x') = -\frac{1}{4\pi}\,\theta[\Sigma(x),x']\,{}_su_{\{A\}}(x,x')\delta\big(\sigma(x,x')\big)\,, \tag{B1}
$$

$$
{}_su_{\{A\}}(x,x') = \begin{cases} \sqrt{\Delta(x,x')} & (s=0), \\ \sqrt{\Delta(x,x')}\bar{g}_{\mu\mu'}(x,x') & (s=1), \\ \sqrt{\Delta(x,x')}\bar{g}_{\mu\mu'}(x,x')\bar{g}_{\nu\nu'}(x,x') & (s=2), \end{cases} \tag{B2}
$$

where the suffix $\{A\}$ stands for the spacetime indices appropriate to the spin s of the field.

From the above Green functions, we obtain the direct part of the field which is expanded around the particle location as

$$
{}_s\phi^{\rm dir}_{\{A\}}(x) = \begin{cases} q\left[\dfrac{1}{\sigma_{;\alpha}(x,z(\tau_{\rm ret}))v^\alpha(\tau_{\rm ret})}\right] + O(y^2)\,, & \text{scalar}\ , \\ e\left[\dfrac{\bar{g}_{\mu\alpha}(x,z(\tau_{\rm ret}))v^\alpha(\tau_{\rm ret})}{\sigma_{;\beta}(x,z(\tau_{\rm ret}))v^\beta(\tau_{\rm ret})}\right] + O(y^2)\,, & \text{vector}\ , \\ 4Gm\left[\dfrac{\bar{g}_{\mu\alpha}(x,z(\tau_{\rm ret}))\bar{g}_{\nu\beta}(x,z(\tau_{\rm ret}))v^\alpha(\tau_{\rm ret})v^\beta(\tau_{\rm ret})}{\sigma_{;\gamma}(x,z(\tau_{\rm ret}))v^\gamma(\tau_{\rm ret})}\right] + O(y^2)\,, & \text{tensor}\ , \end{cases} \tag{B3}
$$

And then, we have

$$
F_\alpha[{}_s\phi^{\rm dir}_{\{A\}}](x) = c\bar{g}_\alpha{}^{\bar\alpha}(x,z_{\rm eq})\frac{1}{\epsilon^3\kappa}\left\{\sigma_{;\bar\alpha}(x,z_{\rm eq}) + h\epsilon^2 R_{\bar\alpha\bar\beta\bar\gamma\bar\delta}(z_{\rm eq})v^{\bar\beta}_{\rm eq}\sigma^{;\bar\gamma}(x,z_{\rm eq})v^{\bar\delta}_{\rm eq}\right\} + O(y)\,, \tag{B4}
$$

$$
\epsilon = \sqrt{2\sigma(x,z_{\rm eq})}\,, \tag{B5}
$$

$$
\begin{aligned}
\kappa &= \sqrt{-\sigma_{\bar\alpha\bar\beta}(x,z_{\rm eq})v^{\bar\alpha}_{\rm eq}v^{\bar\beta}_{\rm eq}} \\
&= 1 + \frac{1}{6}R_{\bar\alpha\bar\beta\bar\gamma\bar\delta}(z_{\rm eq})v^{\bar\alpha}_{\rm eq}\sigma^{;\bar\beta}(x,z_{\rm eq})v^{\bar\gamma}_{\rm eq}\sigma_{;\bar\delta}(x,z_{\rm eq}) + O(y^3)\,,
\end{aligned} \tag{B6}
$$

where c and h depenend on the spin of the field as

$$
(c\,,\ h) = \begin{cases} (q^2\,,\ 1/3)\,, & \text{scalar}\ , \\ (-e^2\,,\ -2/3)\,, & \text{vector}\ , \\ (Gm^2\,,\ -11/3)\,, & \text{tensor}\ . \end{cases} \tag{B7}
$$

Here an extension of the four velocity $v^\alpha(\tau_0)$ necessary to define the projection tensor $P_\alpha{}^\beta$, as mentioned in the line following Eq. (1.3), is chosen such that

$$
V^\alpha(x) = \bar{g}^\alpha{}_{\bar\alpha}(x,z_{\rm eq})v^{\bar\alpha}(\tau_{\rm eq}(x))\,. \tag{B8}
$$

It is noted that when we consider the mode decomposition regularization for the self-force, the direct part calculated in Eqs. (4.26)$\sim$(4.29) is independent of spin.

APPENDIX C: BASIC PROPERTIES OF THE MODE COEFFICIENTS

In this appendix, we examine the general properties of the mode coefficients for the terms that appear in the local coordinate expansion of the direct force given in Eqs. (4.26) $\sim$ (4.29). We show that the C and D-terms of the regularization couter terms vanish in the coincidence limit to the particle position.

We first express ξ in the form,

$$
\xi^2 = \xi_0^2 + \mathcal{L}^2\left(\phi - \phi'\right)^2\ , \tag{C1}
$$

where ξ_0 is defined by

$$
\xi_0 = \sqrt{2}r_0(a - \cos\tilde\theta)^{1/2}\,. \tag{C2}
$$

In terms of ξ_0, R and ϕ, all the terms that contribute to the direct force $F^{\rm dir}_\alpha$ will have the form,

$$
F^{\rm dir}_\alpha \sim \frac{R^p(\phi-\phi')^q\xi_0^{2r}}{\xi_0^3}\left(\frac{R^2}{\xi_0^2}\right)^m\left(\frac{\phi^2}{\xi_0^2}\right)^n\,, \tag{C3}
$$

where we have replaced possible factors of the form Θ^{2k} in favor of polynomials in ξ_0^2, R and ϕ, and m, n, p, q and r are non-negative integers satisfying

$$m \geq 0\,, \quad n \geq 0\,, \quad 1 \leq p+q+2r \leq 3\,. \tag{C4}$$

This is because the highest order of the divergence in the direct force is quadratic, and we may focus only on terms of order up to $O(y^0)$ in the local coordinate expansion.

Let us analyze the coincidence limit in detail. By using a modified version of Eq. (5.4) with $\mathcal{L}^2 = 0$ but by taking account of $O(y^2)$ corrections, one can further reduce the above to the form,

$$(\partial_\phi)^{q+2n} \xi_0^{2q+2n+2r-2m-3} R^{2m+p}\,. \tag{C5}$$

Note that the $O(y^2)$ corrections only change the original q to $q+2$, hence it is enough to consider the above form.

We can decompose (C5) into a spherical harmonic series by using the formula,

$$C_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} \frac{P_\ell(\mu)}{\sqrt{a-\mu}} d\mu = \sqrt{2} \left(a - \sqrt{a^2-1}\right)^{\ell+1/2}\,. \tag{C6}$$

This C_ℓ is equal to the special case of $C_\ell^p(a)$ with $p = -1/2$ given by Eq. (D3). Introducing a variable z by $e^z = a + \sqrt{a^2-1}$ (hence $z > 0$), Eq. (C6) is re-expressed as

$$C_\ell(z) = \frac{2\ell+1}{2} \int_{-1}^{1} \frac{P_\ell(\mu)}{\sqrt{\cosh z - \mu}} d\mu = e^{-Lz}\,, \tag{C7}$$

where $L = \ell + 1/2$. Note that $z \propto |R|$.

As discussed in Section V, odd powers of the operator ∂_ϕ will give the harmonic coefficients which vanish after summing over m. Hence we only need to consider the case of q being an even non-negative integer. Then it is straightforward to see that the operator ∂_ϕ^2 in the harmonic decomposition of (C5) will give the factor L^2 in the coincidence limit of the angular coordinates followed by the sum over m. In general, ∂_ϕ^{2n} will give rise to a polynomial of degree n in L^2. Explicitly, we have

$$\sum_{m=-\ell}^{\ell} m^{2n} \left| Y_{\ell m}\left(\frac{\pi}{2}, 0\right)\right|^2 = \frac{L}{2\pi} \sum_{p=0}^{n} \lambda_p^{(n)} L^{2p}\,. \tag{C8}$$

where the factor $\lambda_p^{(n)}$ is independent of L and $\lambda_n^{(n)} = \Gamma(n+1/2)/(\sqrt{\pi}\,\Gamma(n+1))$. The derivation of this formula is given in Appendix C of Ref. [14].

Thus we may replace $(\partial_\phi)^{q+2n}$ in (C5) by L^{2j} ($1 \leq j \leq q/2+n$; note that q is even). With this replacement, let us consider the following 3 cases for the powers of ξ_0 separately:

(1) $2N+1 := 2m+3-(2q+2r+2n) > 1$ ($\xi_0^{-(2N+1)}$; $N \geq 1$).
In this case, to obtain the harmonic decomposion of $\xi_0^{-(2N+1)}$, one simply applies $[d/d(\cosh z)]^N = [d/\sinh z dz]^N$ to it:

$$\int_{-1}^{1} \frac{P_\ell(\mu)}{(\cosh z - \mu)^{(2n+1)/2}} d\mu \propto \left[\frac{d}{\sinh z dz}\right]^n e^{-Lz}\,. \tag{C9}$$

Taking account of the general form (C5), this will give rise to the harmonic coefficients as

$$\sim R^{2m+p} L^{2j} \frac{L^k}{z^{2N-k}} (1 + \sum_{i\geq 1} c_i z^{2i}) e^{-Lz}, \tag{C10}$$

where $1 \leq k \leq N$. Since $z \sim |R|$ and

$$\begin{aligned} 2m+p &= 2N + 2q + 2r + 2n - 2 + p \\ &\geq 2N + q + (q + 2r + p) - 2 \\ &\geq 2N + q - 1 \geq 2N - 1\,, \end{aligned}$$

the only term that remains in the limit $R \sim \pm z \to 0$ is the $k=1$ term, and this implies $2m+p$ is odd. Since we know the leading divergence is quadratic, this will give a harmonic coefficient proportional to $\mathrm{sign}(R)\, L$ which contributes to A_μ.

(2) $2m+3-(2q+2r+2n)=1$ (ξ_0^{-1}).
In this case, the harmonic coefficents are non-vanishing only if $2m+p=0$. Since the result is independent of L, it contributes to B_μ, i.e., the linearly divergent term.

(3) $2N+1 := 2q+2r+2n-(2m+3) > 1$ (ξ_0^{2N+1}; $N \geq 1$).
In this case, since the harmonic coefficients of ξ_0^{2N+1} will be finite in the limit $z \to 0$, we must have $m=p=0$ which implies $q+2r=2$. Going back to the original form (C5), one can then redefine n by $n+q/2$ and the term of our interest takes the form,

$$\partial_\phi^{2n} \xi_0^{2n-1} \quad (n \geq 1). \tag{C11}$$

Using the formula (D4), we obtain the coefficients of the Legendre decomposition of ξ_0^{2n-1} as

$$\xi_0^{2n-1}\Big|_L = \frac{\kappa_n}{(L^2-1^2)(L^2-2^2)\cdots(L^2-n^2)}\,, \tag{C12}$$

where $\kappa_n = (-1)^n\,[(2n-1)!!]\,r_0^{2n-1}$, and we have introduced the notation,

$$\cdots\Big|_L\,,$$

to denote the $L=\ell+1/2$ mode coefficient of the Legendre expansion in the coincidence limit.[§§] Therefore, together with Eq. (C8), we have

$$\begin{aligned}\partial_\phi^{2n}\xi_0^{2n-1}\Big|_L &= \frac{(-1)^n\kappa_n}{(L^2-1^2)(L^2-2^2)\cdots(L^2-n^2)}\sum_{p=0}^{n}\lambda_p^{(n)}L^{2p}\,,\\ &= (-1)^n\kappa_n\,\lambda_n^{(n)} + \sum_{p=0}^{n-1}\frac{\nu_p}{(L^2-1^2)(L^2-2^2)\cdots(L^2-(n-p)^2)}\,,\end{aligned} \tag{C13}$$

where ν_p is independent of L. We thus find the first term on the right hand side, which is independent of L, contributes to B_μ, while the rest seem to give non-vanishing contributions to D_μ. However, Eq. (C12) tells us that they are simply the Legendre coefficients of positive powers of ξ_0, hence they vanish after the sum over ℓ is taken.

In all of the above three cases, nothing contributes to C_μ. Thus, to summarize, we find the C and D-terms vanish and the A-term is proprotional to sign(R).

APPENDIX D: MATHEMATICAL FORMULAS FOR MODE DECOMPOSITION

In this appendix, we give formulas neccesary for the harmonic decomposition of ξ^{2n-1}. For this purpose, we introduce a dimensionless version of ξ by

$$\tilde{\xi} = \left(a - \cos\tilde{\theta} + \frac{b}{2}(\phi-\phi')^2\right)^{1/2}\,,$$

Here, as defined in the text, $\tilde{\theta}$ is the angle between $\{\theta,\,\phi\}$ and $\{\theta',\,\phi'\}$, and $a>1$. In the following, we do not restrict n to be an integer. Our strategy is to consider a series expansion of $\tilde{\xi}$ by treating b as the expansion parameter.

First, we consider the harmonic expansion of $\tilde{\xi}_0^{2p}$ where $\tilde{\xi}_0 = (a-\cos\tilde{\theta})^{1/2}$,

$$\tilde{\xi}_0^{2p} = (a-\cos\tilde{\theta})^p = 2\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_\ell^p(a) Y_{\ell m}(\theta,\phi)Y_{\ell m}^*(\theta',\phi')\,. \tag{D1}$$

[§§] When the m-sum is non-trivial, $\cdots|_L$ is defined to be the coefficient after summating over m.

The expansion coefficients C_ℓ^p are obtained by the Legendre integration as

$$C_\ell^p(a) = \int_{-1}^{1} d\mu (a-\mu)^p P_\ell(\mu)\,, \tag{D2}$$

where $P_\ell(\mu)$ is the Legendre function of the first kind. As $a > 1$, we can expand $(a-\mu)^p$ in powers of μ. Then we obtain

$$\begin{aligned} C_\ell^p(a) &= \frac{1}{2^{p+1}\Gamma(-p)} a^{p-\ell} \sum_{k=0}^{\infty} \frac{\Gamma(k-p/2+\ell/2)\Gamma(k-p/2+\ell/2+1/2)}{\Gamma(k+\ell+3/2)} \frac{1}{k!}\left(\frac{1}{a^2}\right)^k \\ &= \sqrt{\pi} 2^{-\ell} a^{p-\ell} \frac{\Gamma(-p+\ell)}{\Gamma(-p)\Gamma(\ell+3/2)} F\left(\frac{-p+\ell}{2}, \frac{-p+\ell+1}{2}, \ell+\frac{3}{2}; \frac{1}{a^2}\right) \\ &= -\frac{1}{p+1} a^{-\ell-1}\left(\frac{a^2-1}{2a}\right)^{p+1} F\left(\frac{p}{2}+\frac{\ell}{2}+1, \frac{p}{2}+\frac{\ell}{2}+\frac{3}{2}, p+2; 1-\frac{1}{a^2}\right) \\ &\quad + (-1)^\ell 2^{p+1} a^{p-\ell} \frac{\Gamma(p+1)^2}{\Gamma(p-\ell+1)\Gamma(p+\ell+2)} F\left(-\frac{p}{2}+\frac{\ell}{2}, -\frac{p}{2}+\frac{\ell}{2}+\frac{1}{2}, -p; 1-\frac{1}{a^2}\right). \end{aligned} \tag{D3}$$

In the coincidence limit $a \to 1+0$, we have the two qualitatively different leading behaviors depending on the value of p as

$$C_\ell^p(a) \to \begin{cases} \dfrac{1}{-p-1}(a-1)^{p+1}, & \text{for } p+1<0, \\ (-1)^\ell 2^{p+1} \dfrac{\Gamma(p+1)^2}{\Gamma(p-\ell+1)\Gamma(p+\ell+2)}, & \text{for } p+1>0\,. \end{cases} \tag{D4}$$

It should be noted that the divergent behavior persists in the mode coefficients for $p+1<0$.

Next, we consider the harmonic expansion of $(\phi-\phi')^n(a-\cos\tilde\theta)^p$. We take into account only the leading order behavior in the coincidence limit. Hence we have the basic formula which converts a power of $(\phi-\phi')$ to the ϕ-derivatives,

$$(\phi-\phi')(a-\cos\tilde\theta)^p = \frac{1}{p+1}\frac{\partial}{\partial\phi}(a-\cos\tilde\theta)^{p+1} + O(y^{2p+3})\,. \tag{D5}$$

Using this formula reccursively, we obtain

$$(\phi-\phi')^n(a-\cos\tilde\theta)^p = \sum_{k=0}^{[n/2]} a_k^{(n)} \frac{\partial^{n-2k}}{\partial\phi^{n-2k}}(a-\cos\tilde\theta)^{p+n-k}\,, \tag{D6}$$

where $[n/2]$ denotes the maximum integer not exceeding $n/2$, and $a_k^{(n)}$ satisfies the recurrence relation,

$$a_k^{(n+1)} = \frac{1}{p+n-k+1} a_k^{(n)} - (n-2k+2) a_{k-1}^{(n)}\,, \tag{D7}$$

$$a_k^{(0)} = \begin{cases} 1\,, & k=0\,; \\ 0\,, & k=1,2,3,\cdots. \end{cases} \tag{D8}$$

This is solved to give

$$a_k^{(n)} = \frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(p+n-k+1)\Gamma(n-2k+1)} \frac{(-1)^k}{2^k k!}\,. \tag{D9}$$

Since each ϕ-derivative is interpreted as giving one factor of im to the harmonic coefficients in Eq. (D1), we have the mode decomposition as

$$(\phi-\phi')^n(a-\cos\tilde\theta)^p = 2\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}^{(n,p)}(a) Y_{\ell m}(\theta,\phi) Y_{\ell m}^*(\theta',\phi')\,, \tag{D10}$$

$$C_{\ell m}^{(n,p)}(a) = \sum_{k=0}^{[n/2]} (im)^{n-2k} a_k^{(n)} C_\ell^{p+n-k}(a)\,, \tag{D11}$$

With the formulas (D3) and (D11) at hand, we can now write down the harmonic expansion of $\tilde{\xi}^{2p}$ to the leading order of the local expansion around (θ', ϕ'). The result is

$$\begin{aligned}\tilde{\xi}^{2p} &= \left(a - \cos\tilde{\theta} + \frac{b}{2}(\phi-\phi')^2\right)^p = \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-n+1)} \frac{1}{n!} \left(\frac{b}{2}\right)^n (\phi-\phi')^{2n} (a-\cos\tilde{\theta})^{p-n} \\ &= 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D^p_{\ell m}(a) Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi'),\end{aligned} \tag{D12}$$

where

$$\begin{aligned}D^p_{\ell m}(a) &= \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-n+1)} \frac{1}{n!} \left(\frac{b}{2}\right)^n C^{(2n,p-n)}_{\ell m}(a) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\Gamma(p+1)\Gamma(2n+1)}{\Gamma(p+n-k+1)\Gamma(2n-2k+1)} \frac{(-1)^k b^n}{2^{n+k} n! k!} (-m^2)^{n-k} C^{p+n-k}_{\ell}(a).\end{aligned} \tag{D13}$$

The double sum in the last equation with respect to k and n may be simplified by introducing $\bar{n} := n - k$ and summing over $\bar{n}$ and k, which can be now taken independently. Then we have

$$\begin{aligned}D^p_{\ell m}(a) &= \sum_{\bar{n}=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\Gamma(2\bar{n}+2k+1)}{\Gamma(\bar{n}+k+1)} \frac{1}{k!} \left(\frac{-b}{4}\right)^k \right) \frac{\Gamma(p+1)}{\Gamma(p+\bar{n}+1)\Gamma(2\bar{n}+1)} \left(\frac{-m^2 b}{2}\right)^{\bar{n}} C^{p+\bar{n}}_{\ell}(a) \\ &= \frac{1}{\sqrt{1+b}} \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p+n+1)} \frac{1}{n!} \left(\frac{-m^2 b}{2(1+b)}\right)^n C^{p+n}_{\ell}(a).\end{aligned} \tag{D14}$$

Using Eq. (D4), the leading behavior of the mode coefficients (D14) in the coincidence limit $a \to 1+0$ becomes

$$D^p_{\ell m}(a) \to \begin{cases} \dfrac{1}{\sqrt{1+b}} \dfrac{1}{-p-1} (a-1)^{p+1}, & \text{if } p+1<0; \\ \dfrac{(-1)^\ell 2^{p+1}}{\sqrt{1+b}} \displaystyle\sum_n \frac{\Gamma(p+1)\Gamma(p+n+1)}{\Gamma(p+n-\ell+1)\Gamma(p+n+\ell+2)} \frac{1}{n!} \left(\frac{-m^2 b}{1+b}\right)^n, & \text{if } p+1>0. \end{cases} \tag{D15}$$

Replacing p in the above by $p - 1/2$, we have the formula referred to in the text, Eq. (4.30).

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